

# NON-KÄHLER RICCI FLOW SINGULARITIES THAT CONVERGE TO KÄHLER-RICCI SOLITONS

JAMES ISENBERG, DAN KNOPF, AND NATAŠA ŠEŠUM

**ABSTRACT.** We investigate Riemannian (non-Kähler) Ricci flow solutions that develop finite-time Type-I singularities with the property that parabolic rescalings at the singularities converge to singularity models taking the form of shrinking Kähler-Ricci solitons. More specifically, the singularity models for these solutions are given by the “blowdown soliton” discovered in [FIK03]. Our results support the conjecture that the blowdown soliton is stable under Ricci flow. This work also provides the first set of rigorous examples of non-Kähler solutions of Ricci flow that become asymptotically Kähler, in suitable space-time neighborhoods of developing singularities, at rates that break scaling invariance. These results support the conjectured stability of the subspace of Kähler metrics under Ricci flow.

## CONTENTS

1. Introduction	2
1.1. Behavior of “generic Ricci flow”	2
1.2. Behavior of Ricci flow near Kähler geometries	3
1.3. Organization and main results	4
2. The set-up	4
2.1. Topology and geometry	4
2.2. Ricci flow equations	5
2.3. Closeness Assumptions	6
2.4. Construction of metrics satisfying the Closeness Assumptions	6
3. Characterizing Kähler metrics	7
3.1. The Calabi construction	7
3.2. A coordinate transformation	8
3.3. Ricci flow of Calabi metrics	10
3.4. The blowdown soliton	11
4. Basic estimates	11
4.1. A weak one-sided Kähler stability result	11
4.2. First derivative estimates	12
4.3. Second derivative estimates	13
5. Singularity formation	17
6. Convergence to the blowdown soliton	21
References	23

---

JI thanks the NSF for support in PHY-1306441. DK thanks the NSF for support in DMS-1205270. NŠ thanks the NSF for support in DMS-0905749 and DMS-1056387.

## 1. INTRODUCTION

While the behavior of Ricci flow is fairly well-understood for three-dimensional Riemannian geometries, significantly less is known about four-dimensional Ricci flow. In this work, we study Ricci flow for a certain family of four-dimensional geometries (defined in Section 1.3) that develop finite-time Type-I singularities. The behavior of these solutions illuminates two outstanding issues concerning four-dimensional Ricci flow: i) the stability of certain singularity models in such flows, and ii) the behavior of Ricci flows that start at non-Kähler Riemannian geometries which are nonetheless close to Kähler geometries. To motivate our work here, we discuss each of these issues in turn.

**1.1. Behavior of “generic Ricci flow”.** One of the keys to understanding the nature of singularities that develop in solutions of  $n$ -dimensional Ricci flow is to adequately classify the set of singularity models that may arise. Singularity formation in 3-dimensional Ricci flow has been fairly well-understood since the work of Hamilton [Ham93] and of Perelman [Per02]. Indeed, it follows from the pinching estimate derived by Ivey [Ive93] and improved by Hamilton [Ham93] that the only possible 3-dimensional singularity models have nonnegative sectional curvature, which is a highly restrictive condition. By contrast, Máximo’s results [Max14] imply that, starting in dimension  $n = 4$ , models of finite-time singularity formation can have Ricci curvature of mixed sign (even for Kähler solutions). This leaves nonnegative scalar curvature as the only known restriction on singularity models for  $n \geq 4$ . This restriction is too weak to be useful.

In dimensions  $n \geq 4$ , therefore, a classification of all singularity models is impractical. A more promising alternative is to try to classify those models that are generic, or at least stable. A singularity model developing from certain original data is labeled *stable* if flows starting from all sufficiently small perturbations of that data develop singularities with the same singularity model; it is labeled *generic* if flows that start from an open dense subset of all possible initial data develop singularities having the same singularity model. Clearly, a singularity model can be generic only if it is stable.

Important work of Colding and Minicozzi (see [CM12] and [CM15]) provides strong support in favor of the conjecture that the only generic singularities of Mean Curvature Flow are generalized cylinders  $\mathbb{R}^m \times \mathbb{S}^{n-m}$ . Although no analogous result is currently known for Ricci flow, a conjectural picture comes from the work of Cao, Hamilton, and Ilmanen [CHI04], who define the *central density*  $\Theta$  and the *entropy*  $\nu(\mathcal{M})$  of a shrinking Ricci soliton  $\mathcal{M}$ , using Perelman’s reduced volume and entropy, respectively (see [Per02]). They observe that their central density imposes a partial order on shrinking solitons: monotonicity of the  $\nu$ -functional in time means that if perturbations of a shrinking soliton develop singularities, these cannot be modeled on solitons of lower density. (Compare [CM12].)

Motivated partly by [CHI04], it is conjectured by experts (see, e.g., [HHS14]) that the only candidates to be generic singularity models in dimension  $n = 4$  are  $\mathbb{S}^4$ ,  $\mathbb{S}^3 \times \mathbb{R}$ ,  $\mathbb{S}^2 \times \mathbb{R}^2$ , and  $\mathbb{CP}^2$  (all with their canonical metrics), and  $(\mathcal{L}_{-1}^2, h)$ . (Although the Fubini–Study metric on  $\mathbb{CP}^n$  is well known to be weakly linearly stable, an argument for its dynamic instability is presented Kröncke [Kro13].) The manifold  $(\mathcal{L}_{-1}^2, h)$ , which is constructed and studied in [FIK03], is a  $U(2)$ -invariant

gradient Kähler shrinker on the complex line bundle<sup>1</sup>  $\mathbb{C} \hookrightarrow \mathcal{L}_{-1}^2 \rightarrow \mathbb{CP}^1$ , which is the complex bundle  $\mathcal{O}(-1)$ ; i.e., it is the blow-up of  $\mathbb{C}^2$  at the origin. (We note that Perelman's  $\nu$  functional is not defined on the noncompact manifold  $(\mathcal{L}_{-1}^2, h)$ . This is one of the reasons why the claim that the list above includes all 4-dimensional generic singularity models is conjectural.)

While the construction of the  $(\mathcal{L}_{-1}^2, h)$  shrinker involves the blowup of a point on  $\mathbb{C}^2$ , following the authors of [CHI04], we call  $(\mathcal{L}_{-1}^2, h)$  the *blowdown soliton*. We do this because, as shown in Theorem 1.6 of [FIK03], there is a family of Riemannian manifolds  $\mathcal{N}_t$ ,  $-\infty < t < \infty$ , with the following features: for  $t < 0$ ,  $\mathcal{N}_t$  is  $(\mathcal{L}_{-1}^2, h(t))$ ; for  $t = 0$ ,  $\mathcal{N}_0$  is a Kähler cone on  $\mathbb{C}^2$  with an isolated singularity at the origin; and for  $t > 0$ ,  $\mathcal{N}_t$  is an expanding soliton discovered by Cao [Cao97]. It is expected that according to most (if not all) of the definitions of a *weak solution of Ricci flow* which are currently being explored (e.g., see [HN15] and [Stu16]), the family  $\mathcal{N}_t$  will qualify for such a designation. Consequently, in this weak sense, one sees that Ricci flow can carry out an algebraic-geometric blowdown.

As noted above, a pre-condition for a singularity model being generic is that it must be a stable attractor for Ricci flow — regarded as a dynamical system on the space of Riemannian metrics. Stability of  $S^4$  is well established. (In fact, Brendle and Schoen [BS09] show that its basin of attraction includes all 1/4-pinched metrics.) Stability of generalized cylinders is strongly conjectured but not known for Ricci flow. Stability of the blowdown soliton is also not known, although Máximo's proof [Max14] shows that arbitrarily small  $U(2)$ -invariant Kähler perturbations of the shrinking soliton on  $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$  (which was discovered independently by Koiso [Koi90] and by Cao [Cao96]) develop singularities modeled on  $(\mathcal{L}_{-1}^2, h)$ . (We remark that prior to Máximo's results, it was shown in [HM11] that the Koiso-Cao soliton is *linearly* unstable.)

Our results in this paper prove that the blowdown soliton is a singularity model attractor for solutions of Ricci flow that originate from a set of Riemannian initial data defined by a structural (isometry) hypothesis and by a (weak) set of pinching conditions that we specify in Section 2 below. Metrics in this set are not Kähler. These results provide evidence in favor of the conjectured stability of  $(\mathcal{L}_{-1}^2, h)$ .

**1.2. Behavior of Ricci flow near Kähler geometries.** As noted above, the  $(\mathcal{L}_{-1}^2, h)$  shrinker is Kähler. Hence, the study of non-Kähler Ricci flows near the blowdown soliton provides information about the difficult issue of the behavior of Ricci flow solutions that start near, but not in, the subspace of Kähler metrics. Do those solutions stay near or (better) asymptotically approach that subspace, which is of infinite codimension? It is believed by many experts that the subspace of Kähler metrics *should* be dynamically stable for nearby solutions of Ricci flow. Evidence in favor of this conjecture is provided by the work of Streets and Tian [ST11], who prove that the Kähler subspace is an attractor for Hermitian curvature flow.

While we do not establish a general stability principle for Kähler geometries, the results of this paper do provide the first rigorous examples of non-Kähler solutions of Ricci flow that become asymptotically Kähler, in suitable space-time neighborhoods of developing singularities, at rates that break scaling invariance. We hope that these examples provide motivation for further study of the general question, particularly in (real) dimension  $n = 4$ .

<sup>1</sup>The bundle we label  $\mathcal{L}_{-1}^2$  here is denoted by  $\mathcal{L}^{-1}$  in [FIK03] and by  $L(2, -1)$  in [CHI04].

**1.3. Organization and main results.** The general class of Riemannian geometries that we study in this paper are smooth cohomogeneity-one metrics on the closed manifold  $\mathbb{S}^2 \tilde{\times} \mathbb{S}^2$  (the “twisted bundle” of  $\mathbb{S}^2$  over  $\mathbb{S}^2$ ). We describe these geometries (which we label “[ $\mathbb{S}^2 \tilde{\times} \mathbb{S}^2$ ]-warped Berger geometries”) in detail below in Section 2. Here, for the purposes of stating our main theorem, we note that for these metrics, there are two distinguished fibers  $\mathbb{S}^2_{\pm}$  (at either “pole”); by contrast, a generic fiber is diffeomorphic to  $\mathbb{S}^3$ .

In Section 2.3, we identify an open subset of the [ $\mathbb{S}^2 \tilde{\times} \mathbb{S}^2$ ]-warped Berger geometries by means of five pinching inequalities. These inequalities constitute our Closeness Assumptions, which we require the initial data for our Ricci flow solutions to satisfy. These assumptions ensure that our initial data, while not Kähler, are “not too far” from the subspace of Kähler metrics. In Section 2.4, we prove that our assumptions are not vacuous; i.e., we show that the open subset of initial data satisfying the Closeness Assumptions is not empty.

We clarify the relationship between Kähler geometries and the [ $\mathbb{S}^2 \tilde{\times} \mathbb{S}^2$ ]-warped Berger geometries in Section 3. Also in that section, we provide some background information about the blowdown soliton.

In the remainder of this paper, we prove a sequence of Lemmata and Corollaries that combine to establish the following result:

**Main Theorem.** *There exists a nonempty open set of non-Kähler metrics on  $\mathbb{S}^2 \tilde{\times} \mathbb{S}^2$  (contained in the [ $\mathbb{S}^2 \tilde{\times} \mathbb{S}^2$ ]-warped Berger class, and satisfying the Closeness Assumptions) such that any Ricci flow solution originating from this set has the following properties:*

- (1) *Inequalities (a)–(d) in the Closeness Assumptions are preserved by the flow.*
- (2) *The solution develops a Type-I singularity at  $T < \infty$ , with  $|\mathbb{S}^2_-(T)| = 0$ .*
- (3) *Every blow-up sequence  $(\mathbb{S}^2 \tilde{\times} \mathbb{S}^2, G_k(t), p)$  with  $p \in \mathbb{S}^2_-$  subconverges to a Kähler singularity model that is the blowdown shrinking soliton  $(\mathcal{L}^2_{-1}, h)$ .*

## 2. THE SET-UP

**2.1. Topology and geometry.** In [IKS16], we study “warped Berger” metrics which take the form

$$(1) \quad G = ds \otimes ds + \left\{ f^2 \omega^1 \otimes \omega^1 + g^2 (\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3) \right\}$$

on  $[s_-, s_+] \times \text{SU}(2)$ , where  $\{\omega^1, \omega^2, \omega^3\}$  constitutes a one-form basis for  $\text{SU}(2)$ , where  $s(x, t)$  denotes arclength from  $x = 0$ , with  $x \in [-1, 1]$ , and where we set  $s_{\pm} := s(\pm 1)$ . The functions  $f$  and  $g$  depend only on  $x$  (or equivalently on  $s$ ); hence these metrics are cohomogeneity one. In [IKS16], we choose boundary conditions on  $f$  and  $g$  that result in these metrics inducing geometries on  $\mathbb{S}^3 \times \mathbb{S}^1$ . Here, we instead choose boundary conditions on  $f$  and  $g$  that result in smooth cohomogeneity geometries on  $\mathbb{S}^2 \tilde{\times} \mathbb{S}^2$ , thereby defining the class of [ $\mathbb{S}^2 \tilde{\times} \mathbb{S}^2$ ]-warped Berger geometries. We do this as follows.

It is a standard result in Riemannian geometry that one may smoothly close the boundary at  $s_-$ , provided that the functions  $f_-(s) := f(s_- + s)$  and  $g_-(s) := g(s_- + s)$  defined for  $0 \leq s \leq s_+ - s_-$  satisfy

$$(2) \quad f_-^{(\text{even})}(0) = 0, \quad f'_-(0) = 1, \quad \text{and} \quad g_-(0) > 0, \quad g_-^{(\text{odd})}(0) = 0.$$

The topology then locally becomes that of the disc bundle  $\mathcal{D}^2 \hookrightarrow \mathcal{D}_1^4 \twoheadrightarrow \mathbb{S}^2$  with Euler class 1 and boundary  $\partial\mathcal{D}_1^4 \approx \mathbb{S}^3$  that appears in the handlebody construction of  $\mathbb{CP}^2$ . Note that the 2-sphere here is the base of the Hopf fibration on  $\mathbb{S}^3 \approx \text{SU}(2)$ . If one repeats this construction at  $s_+$ , with  $f_+(s) := f(s_+ + s)$  and  $g_+(s) := g(s_+ + s)$  defined for  $s_- - s_+ \leq s \leq 0$  satisfying

$$(3) \quad f_+^{(\text{even})}(0) = 0, \quad f'_+(0) = -1, \quad \text{and} \quad g_+(0) > 0, \quad g_+^{(\text{odd})}(0) = 0,$$

one obtains a closed 4-manifold with the topology of  $\mathbb{S}^2 \tilde{\times} \mathbb{S}^2$ . We denote by  $\mathbb{S}_\pm^2$  the distinguished 2-spheres that appear as the base spaces in the closing construction at either “pole”  $s_\pm$ . We note that while  $\mathbb{S}^2 \tilde{\times} \mathbb{S}^2$  is diffeomorphic to  $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ , the Ricci flow evolutions we study are not Kähler.

The metrics  $G = ds \otimes ds + f^2 \omega^1 \otimes \omega^1 + g^2 \omega^2 \otimes \omega^2 + h^2 \omega^3 \otimes \omega^3$  described in Appendix A of [IKS16] are clearly  $\text{SU}(2)$ -invariant. The simplifying assumption  $h \equiv g$  made here enlarges their symmetry group to  $\text{U}(2)$ . However, although  $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$  admits Kähler metrics, including the  $\text{U}(2)$ -invariant Kähler–Ricci soliton mentioned above, we observe in Lemma 1 that metrics of the form (1) cannot be Kähler unless they satisfy the closed condition  $f = gg_s$ .

**2.2. Ricci flow equations.** We study in this work solutions  $(\mathbb{S}^2 \tilde{\times} \mathbb{S}^2, G(t))$  of Ricci flow that originate from smooth initial data  $G(0)$  satisfying the closing conditions (2) and (3) for the  $[\mathbb{S}^2 \tilde{\times} \mathbb{S}^2]$ -warped Berger geometries, as outlined above. For as long as such solutions remain smooth, the functions  $f$  and  $g$  continue to satisfy conditions (2) and (3), and hence remain  $[\mathbb{S}^2 \tilde{\times} \mathbb{S}^2]$ -warped Berger geometries.

Since the metrics studied in [IKS16] and those studied here are the same apart from boundary conditions, we may use formulas (10)–(13) of [IKS16] to obtain the sectional curvatures of the metric<sup>2</sup>  $G$ :

$$(4a) \quad \kappa_{12} = \kappa_{31} = \frac{f^2}{g^4} - \frac{f_s g_s}{f g},$$

$$(4b) \quad \kappa_{23} = \frac{4g^2 - 3f^2}{g^4} - \frac{g_s^2}{g^2},$$

$$(4c) \quad \kappa_{01} = -\frac{f_{ss}}{f},$$

$$(4d) \quad \kappa_{02} = \kappa_{03} = -\frac{g_{ss}}{g}.$$

Writing the metric in coordinate form (1), we note that its evolution under Ricci flow is governed by the evolution equations for  $f$  and  $g$ , which (as shown in (14) of [IKS16]) take the following form:

$$(5a) \quad f_t = f_{ss} + 2\frac{g_s}{g}f_s - 2\frac{f^3}{g^4},$$

$$(5b) \quad g_t = g_{ss} + \left(\frac{f_s}{f} + \frac{g_s}{g}\right)g_s + 2\frac{f^2 - 2g^2}{g^3}.$$

The variable  $s = s(x, t)$ , representing arclength from the  $\mathbb{S}^3$  at  $x = 0$ , is a choice of gauge that results in this system being manifestly strictly parabolic. The cost one

<sup>2</sup>Using L'Hôpital's rule, it is straightforward to verify that all quantities appearing in this section are well defined at  $\mathbb{S}_\pm^2$ . We make this explicit below.

pays for this is the non-vanishing commutator,

$$(6) \quad \left[ \frac{\partial}{\partial t}, \frac{\partial}{\partial s} \right] = -(\log \rho)_t \frac{\partial}{\partial s} = - \left( \frac{f_{ss}}{f} + 2 \frac{g_{ss}}{g} \right) \frac{\partial}{\partial s}.$$

**2.3. Closeness Assumptions.** The Riemannian Ricci flow solutions we study here originate from an open set of cohomogeneity-one metrics that is defined by certain mild hypotheses, which effectively guarantee that at least initially, the metrics are “somewhat close” to the subspace of Kähler metrics.

*Closeness Assumptions.* At time  $t = 0$ , the metric  $G$  of the form (1) determined by the pair  $(f, g)$  satisfies the following:

- (a)  $f \leq g$ ;
- (b)  $g|g_s| \leq f$ ;
- (c)  $|f_s| \leq 2/\sqrt{3}$ ;
- (d)  $g^2(s_+) - 3g^2(s_-) \geq \delta^2$  for some  $\delta > 0$ ; and
- (e)  $g_s \geq 0$ , with strict inequality off  $\mathcal{S}_\pm^2$ .

It follows from Lemma 26 of [IKS16] that condition (a) is preserved under the flow. We prove in Section 4.2 that condition (b) — which, as we show there, may be regarded as a “Kähler pinching condition” — and condition (c) are preserved by the flow. We prove in Section 5 that (d) is preserved. We do not need (e) to be preserved; we need it only for initial comparison with a barrier function constructed in Section 5.

**Remark 1.** *Regarding our Closeness Assumptions, we note the following: i) Even for Kähler metrics, condition (d) is necessary so that the  $g^2(\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3)$  factor in the metric  $G$  vanishes before the  $(ds \otimes ds + f^2 \omega^1 \otimes \omega^1)$  factor does; this is necessary for the singularities to form on  $\mathcal{S}_-^2$  (see Theorem 1.1 of [SW11] and Remark 4 below). ii) Although we cannot prove that monotonicity of  $g$  is preserved for non-Kähler metrics satisfying our Closeness Assumptions, we use condition (e) to prove that for any such metric,  $g$  remains above a subsolution we construct that does remain strictly monotone in space.*

**2.4. Construction of metrics satisfying the Closeness Assumptions.** We choose  $f$  to be any smooth function that is defined for  $s \in [s_-, s_+]$ , is strictly positive except at  $s_\pm$ , satisfies  $|f_s| \leq 1$  with equality only at  $s_\pm$ , and satisfies the closing conditions (2) and (3). For each such function, we now construct an infinite-dimensional family  $\mathcal{G}_{\alpha, \delta, \varepsilon}$  of initial metrics which satisfy our Closeness Assumptions. The family depends on parameters  $\alpha, \delta$ , and  $\varepsilon$ , to be chosen below. We define

$$A^2 := 2 \int_{s_-}^{s_+} f(s) ds,$$

noting that we are free to let the difference  $s_+ - s_-$ , and hence  $A^2$ , be as large as we wish. We then choose  $\alpha$  and  $\delta$  to be any positive parameters satisfying

$$(7) \quad \alpha^2 + \delta^2 \leq \frac{A^2}{2}.$$

To define  $g$ , and hence a metric  $(f, g) \in \mathcal{G}_{\alpha, \delta, \varepsilon}$ , we choose  $\varphi$  to be any smooth function satisfying  $1 - \varepsilon \leq \varphi \leq 1$ , requiring that it be nonconstant unless  $\varepsilon = 0$ .

Clearly  $\varepsilon$  controls how much  $\varphi$  can stray from being constant. We then set

$$g^2(s) := \alpha^2 + 2 \int_{s_-}^s \varphi(\bar{s}) f(\bar{s}) d\bar{s}.$$

We readily verify that  $g$  defined in this way satisfies closing conditions (2) and (3).

To verify that part (a) of our Closeness Assumptions is satisfied, we observe that the gradient restriction  $|f_s| \leq 1$  implies that

$$\begin{aligned} f^2(s) &= 2 \int_{s_-}^s f(\bar{s}) f_{\bar{s}}(\bar{s}) d\bar{s} \\ &\leq 2 \int_{s_-}^s \{\varphi + (1 - \varphi)\} f(\bar{s}) d\bar{s} \\ &\leq g^2 - \alpha^2 + \varepsilon A^2 \\ &\leq g^2, \end{aligned}$$

provided that

$$(8) \quad \varepsilon \leq \frac{\alpha^2}{A^2}.$$

It immediately follows that part (b) holds for all  $\varepsilon \in [0, 1)$ , with equality — which Lemma 1 (below) shows is equivalent to the metric being Kähler — if and only if  $\varepsilon = 0$ . Part (c) holds as a consequence of the gradient restriction  $|f_s| \leq 1$ .

To verify that part (d) holds, we observe that one has

$$\begin{aligned} g^2(s_+) - 3g^2(s_-) &\geq \{\alpha^2 + (1 - \varepsilon)A^2\} - 3\alpha^2 \\ &= (A^2 - 2\alpha^2) - \varepsilon A^2 \\ &\geq 2\delta^2 - \varepsilon A^2, \end{aligned}$$

with the last inequality following from the restrictions on  $\alpha$  and  $\delta$  that we have imposed in (7). Hence we satisfy part (d) so long as

$$(9) \quad \varepsilon \leq \frac{\delta^2}{A^2}.$$

Finally, it is clear that part (e) is satisfied, because  $gg_s \geq (1 - \varepsilon)f \geq 0$ .

### 3. CHARACTERIZING KÄHLER METRICS

We prove in this paper that as they become singular, solutions originating from initial data satisfying our Closeness Assumptions asymptotically approach the blow-down soliton. For the reader's convenience, we include here a brief review of metrics related to that singularity model.

**3.1. The Calabi construction.** We call a metric on  $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$  or  $\mathcal{L}_{-1}^2$  a *Calabi metric* if it is Kähler and U(2)-invariant. As part of a much more general construction [Cal82], Calabi has observed that any U(2)-invariant Kähler metric on  $\mathbb{C}^2 \setminus (0, 0)$  has the form

$$(10) \quad h_{\mathbb{C}^2 \setminus (0,0)} = \left\{ e^{-r} \phi \delta_{\alpha\beta} + e^{-2r} (\phi_r - \phi) \bar{z}^\alpha z^\beta \right\} dz^\alpha \otimes d\bar{z}^\beta.$$

Here  $r := \log(|z_1|^2 + |z_2|^2)$  is Calabi's coordinate, and  $\phi(r) = P_r(r)$ , where  $P$  is the Kähler potential. The metric closes smoothly at the origin, hence induces a

smooth metric on the total space of the bundle  $\mathcal{L}_{-1}^2$  (or on a neighborhood of  $\mathcal{S}_{-}^2$  in  $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ ) if and only if there are  $a_0, a_1 > 0$  such that

$$(11) \quad \phi(r) = a_0 + a_1 e^r + a_2 e^{2r} + \mathcal{O}(e^{3r}) \quad \text{as } r \rightarrow -\infty.$$

The metric closes smoothly at spatial infinity, hence induces a smooth Kähler metric with respect to the unique complex structure on  $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ , if and only if two conditions hold: i)  $\phi_r > 0$  everywhere, and ii) there are  $b_0 > a_0$  and  $b_1 < 0$  such that

$$\phi(r) = b_0 + b_1 e^{-r} + b_2 e^{-2r} + \mathcal{O}(e^{-3r}) \quad \text{as } r \rightarrow \infty.$$

Alternatively, one may obtain a complete Calabi metric on the noncompact space  $\mathcal{L}_{-1}^2$  by imposing conditions at spatial infinity that guarantee completeness; see, e.g., [FIK03]. As noted in equation (19) of that paper, any  $U(2)$ -invariant metric on  $\mathbb{C}^2 \setminus (0, 0)$  can be written in real coordinates on  $\mathbb{R}^4 \setminus (0, 0, 0, 0)$  as

$$(12) \quad h_{\mathbb{R}^4 \setminus (0,0,0,0)} = \phi_r \left( \frac{1}{4} dr \otimes dr + \omega^1 \otimes \omega^1 \right) + \phi (\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3).$$

**3.2. A coordinate transformation.** A comparison of equations (1) and (12) shows that a coordinate transformation is needed to write a Calabi metric in the  $s$ -coordinate system. We implement this as follows. Recalling that  $s(x, t)$  denotes arclength from the  $\mathcal{S}^3$  at the “interior” point  $x = 0$ , and motivated by Calabi’s (fixed)  $r$ -coordinate introduced in Section 3.1, we define here a function

$$(13) \quad \rho(s, t) := 2 \int_0^s \frac{d\bar{s}}{f(\bar{s}, t)}.$$

The closing conditions then show that  $\rho \rightarrow \pm\infty$  at  $\mathcal{S}_{\pm}^2$ . Moreover, one has

$$(14) \quad ds = \frac{1}{2} f d\rho,$$

so that equation (1) may be re-expressed in the form

$$(15) \quad G = f^2 \left( \frac{1}{4} d\rho \otimes d\rho + \omega^1 \otimes \omega^1 \right) + g^2 (\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3),$$

where we emphasize that the coordinate  $\rho$  is allowed to depend on time. We note that  $\rho$  and its time evolution depend only on  $s(x, t)$  and  $f(s(x, t), t)$ , neither of which depend on  $\rho$ .

We observe that equation (15) has the form (12) of a Calabi metric  $h$  if and only if  $g^2 = \phi$  and  $f^2 = (g^2)_{\rho}$ , in which case one has

$$f = gg_s \quad \text{and} \quad f_s = gg_{ss} + g_s^2.$$

We summarize this simple observation, which is crucial to our work here, as follows:

**Lemma 1.** *A  $[\mathcal{S}^2 \tilde{\times} \mathcal{S}^2]$ -warped Berger metric (1) is Kähler if and only if  $f = gg_s$ .*



If  $G$  is Kähler, then its sectional curvatures (which generally take the form (4)) take the following special form:

$$\begin{aligned}\kappa_{12} &= \kappa_{31} = \kappa_{02} = \kappa_{03} = -\frac{g_{ss}}{g}, \\ \kappa_{23} &= 4\frac{1-g_s^2}{g^2}, \\ \kappa_{01} &= -\frac{g_{sss}}{g_s} - 3\frac{g_{ss}}{g}.\end{aligned}$$

As must be true for a Kähler metric on a complex surface, the Ricci endomorphism then has only two eigenvalues,

$$R_0^0 = R_1^1 = -\frac{g_{sss}}{g_s} - 5\frac{g_{ss}}{g} \quad \text{and} \quad R_2^2 = R_3^3 = -2\frac{g_{ss}}{g} + 4\frac{1-g_s^2}{g^2}.$$

Because Kähler–Ricci flow is strictly parabolic, no time-dependent choice of gauge  $s(x, t)$  is needed to ensure parabolicity. Rather, one can write the Kähler–Ricci PDE with respect to a time-independent coordinate. The following observation is a particular instance of this general fact.

**Lemma 2.** *The evolution equation for the coordinate  $\rho$  under Ricci flow takes the form*

$$(16) \quad \rho_t = 2 \int_0^\rho \left\{ \frac{g_{ss}}{g} - \frac{f_s g_s}{f g} + \frac{f^2}{g^4} \right\} d\bar{\rho}.$$

*For a Kähler geometry, the integrand in (16) vanishes pointwise; hence, for Kähler initial data, the coordinate  $\rho$  is independent of  $t$ .*

**Remark 2.** *For Kähler initial data, one may therefore assume without loss of generality that  $\rho$  is identical to Calabi’s coordinate  $r = \log(|z_1|^2 + |z_2|^2)$ .*

*Proof of Lemma 2.* It follows from equation (56) in [IKS16] and from (4) above that the gauge quantity  $\frac{\partial s}{\partial x}$  evolves according to the equation

$$\left( \frac{\partial s}{\partial x} \right)_t = \left\{ \frac{f_{ss}}{f} + 2\frac{g_{ss}}{g} \right\} \frac{\partial s}{\partial x}.$$

Hence, using equations (13) and (5a), we determine that the time derivative of  $\rho$  at fixed  $x$  is given by

$$\begin{aligned}\frac{1}{2}\rho_t &= \frac{\partial}{\partial t} \left( \int_0^x f^{-1}(s(\bar{x}, t), t) \frac{\partial s}{\partial \bar{x}} d\bar{x} \right) \\ &= \int_0^x \left\{ f^{-1} \left( \frac{\partial s}{\partial \bar{x}} \right)_t - f^{-2} f_t \left( \frac{\partial s}{\partial \bar{x}} \right) \right\} d\bar{x} \\ &= 2 \int_0^s \left\{ \frac{g_{\bar{s}\bar{s}}}{f g} - \frac{f_{\bar{s}} g_{\bar{s}}}{f^2 g} + \frac{f}{g^4} \right\} d\bar{s}.\end{aligned}$$

This proves the first claim. The second follows by direct computation.  $\square$

**3.3. Ricci flow of Calabi metrics.** Lemma 1 states that the initial metric is Calabi if and only if  $f = gg_s$ . Because Ricci flow preserves the Kähler condition with respect to the original complex structure (here, the unique complex structure on  $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ ) and also preserves initial symmetries, a solution originating from Calabi initial data remains Calabi for as long as it exists. This can be seen directly, as we now observe.

In this section, which is not needed for the rest of the paper, we find it convenient to work with  $u := f^2$  and  $v := g^2$ . The Ricci flow evolution equations for these quantities are given by

$$\begin{aligned} u_t &= u_{ss} - \frac{u_s^2}{2u} + \frac{u_s v_s}{v} - 4 \frac{u^2}{v^2}, \\ v_t &= v_{ss} + \frac{u_s v_s}{2u} + 4 \frac{u - 2v}{v}. \end{aligned}$$

On a Calabi solution, one can use the relation  $u = v_s^2/4$  (equivalent to  $f = gg_s$ ) to simplify the evolution equation above for  $v$ , thereby obtaining

$$(17) \quad v_t = 2v_{ss} + \frac{v_s^2}{v} - 8.$$

One now has two ways of computing the evolution of  $u$ . Evaluating the equation above for  $u_t$  by using the Kähler condition  $u = v_s^2/4$  to convert the RHS into terms involving only  $v$  and its derivatives, one obtains

$$(18) \quad u_t = \frac{1}{2} v_s v_{sss} + \frac{1}{2} \frac{v_{ss} v_s^2}{v} - \frac{1}{4} \frac{v_s^4}{v^2}.$$

On the other hand, one can differentiate the RHS of  $u = v_s^2/4$  directly, use the commutator  $[\partial_t, \partial_s]$  given in equation (6), and then apply (17), obtaining

$$\begin{aligned} \left( \frac{v_s^2}{4} \right)_t &= \frac{1}{2} v_s (v_s)_t \\ &= \frac{1}{2} v_s \left\{ (v_t)_s - \left( \frac{f_{ss}}{f} + 2 \frac{g_{ss}}{g} \right) v_s \right\} \\ &= \frac{1}{2} v_s v_{sss} + \frac{1}{2} \frac{v_{ss} v_s^2}{v} - \frac{1}{4} \frac{v_s^4}{v^2}, \end{aligned}$$

as above. This calculation verifies directly what one knows from general principles: that the Calabi condition is preserved by the flow. We note in particular that for a Calabi solution, the Ricci flow system reduces to a scalar PDE, in the sense that the evolution of  $u$  is completely determined by the evolution of  $v$ .

**Remark 3.** For solutions with initial data satisfying our Closeness Assumptions, the fact that  $g_s > 0$  everywhere except at  $S_\pm^2$  holds initially. For as long as this remains true (presumably only a short time for non-Kähler solutions), there is a well-defined function  $\theta$  such that

$$f = \theta gg_s.$$

We note that  $\theta \equiv 1$  for a Kähler solution, and that the evolution equation for  $\theta$  is

$$\theta_t = \theta_{ss} + 2 \frac{f g_s - 2 f_s g}{g^2} (\theta^2 - 1),$$

which yields an easy direct proof that the Kähler condition is preserved for these geometries.

**3.4. The blowdown soliton.** Under Kähler–Ricci flow, the evolution of an arbitrary Calabi metric

$$(19) \quad h = \left\{ e^{-r} \phi \delta_{\alpha\beta} + e^{-2r} (\phi_r - \phi) \bar{z}^\alpha z^\beta \right\} dz^\alpha \otimes d\bar{z}^\beta,$$

written in terms of Calabi’s fixed  $r$ -coordinate on  $\mathcal{L}_{-1}^2$  or on  $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ , is determined by the PDE

$$(20) \quad \phi_t = \frac{\phi_{rr}}{\phi_r} + \frac{\phi_r}{\phi} - 2.$$

The blowdown soliton is specified by setting  $\phi$  in (19) equal to a function  $\varphi$  which (following Lemma 6.1 and equation (27) of [FIK03] with  $\lambda = -1$ ,  $\mu = \sqrt{2}$ , and  $\nu = 0$ ) satisfies the separable first-order ODE

$$(21) \quad \varphi_r = \frac{1}{\sqrt{2}} \varphi - (\sqrt{2} - 1) - \left( 1 - \frac{1}{\sqrt{2}} \right) \varphi^{-1}.$$

Rewriting this ODE in the form

$$dr = \frac{\varphi d\varphi}{\varphi - 1} - \frac{\varphi d\varphi}{\varphi + \sqrt{2} - 1},$$

one can solve it implicitly, obtaining

$$(22) \quad e^{r+\chi} = \frac{\varphi - 1}{(\varphi + \sqrt{2} - 1)^{\sqrt{2}-1}}.$$

The arbitrary constant  $\chi$  above reflects the fact that the soliton is unique only modulo translations in  $r$ . Examination of formula (22) also shows that the soliton is cone-like at spatial infinity and hence complete.

Equation (24) of [FIK03] implies that the blowdown soliton function  $\varphi$  also satisfies the second-order ODE

$$(23) \quad \frac{\varphi_{rr}}{\varphi_r} + \frac{\varphi_r}{\varphi} - \sqrt{2} \varphi_r + \varphi - 2 = 0.$$

It follows from (23) that  $\varphi$  satisfies  $\varphi_t = \sqrt{2} \varphi_r - \varphi$ ; hence the soliton evolves by translation and scaling.

#### 4. BASIC ESTIMATES

In this section, we prove a number of estimates that are needed for the proof of our Main Theorem.

**4.1. A weak one-sided Kähler stability result.** We begin by introducing the useful quantity

$$(24) \quad \psi := \left( \frac{g g_s}{f} \right)^2 - 1.$$

This quantity  $\psi$  is well defined at  $\mathbb{S}_\pm^2$ , because it follows from l’Hôpital’s rule that  $\frac{g g_s}{f} \big|_{\mathbb{S}_\pm^2} = g g_{ss}$ .

Lemma 1 tells us that  $\psi \equiv 0$  if and only if the metric  $G$  from (1) is Kähler. Therefore, we use  $\psi$  to measure, in a precise sense, how far away a solution is from being Kähler. The following result is thus a statement of weak (one-sided) stability for the Kähler condition. Note that part (b) of our Closeness Assumptions ensures that  $\psi \leq 0$  at  $t = 0$ .

**Lemma 3.** *If  $-1 \leq \psi \leq 0$  initially, then  $-1 \leq \psi \leq 0$  as long as the flow exists.*

*Proof.* It is only necessary to prove the upper bound. The quantity  $\psi$  evolves under Ricci flow by

$$(25) \quad \psi_t = \psi_{ss} + \left\{ 3 \frac{f_s}{f} - 2 \frac{g_s}{g} \right\} \psi_s - \frac{\psi_s^2}{2(\psi+1)} + \left\{ 4 \frac{g_s^2}{g^2} - 8 \frac{f_s g_s}{f g} \right\} \psi.$$

From this equation, it is clear that the condition  $\psi \leq 0$  is preserved if all maxima of  $\psi$  occur away from  $\mathcal{S}_{\pm}^2$ .

If a maximum occurs instead at  $\mathcal{S}_{\pm}^2$ , then we apply l'Hôpital to determine that

$$\frac{f_s \psi_s}{f} \Big|_{\mathcal{S}_{\pm}^2} = \psi_{ss} \quad \text{and} \quad \frac{f_s g_s}{f g} \Big|_{\mathcal{S}_{\pm}^2} = \frac{g_{ss}}{g}.$$

Hence

$$\psi_t \Big|_{\mathcal{S}_{\pm}^2} = 4\psi_{ss} - 8 \frac{g_{ss}}{g} \psi.$$

However, smoothness of either function  $\psi_{\pm}(s, \cdot) := \psi(s - s_{\pm}, \cdot)$  at a maximum on  $\mathcal{S}_{\pm}^2$  shows that  $\psi_{ss} \Big|_{\mathcal{S}_{\pm}^2} = (\psi_{\pm})_{ss} \Big|_{\mathcal{S}_{\pm}^2} \leq 0$ . The result follows.  $\square$

**4.2. First derivative estimates.** Based on the one-sided Kähler stability established in Lemma 3, we now derive estimates on the first derivatives of  $f$  and  $g$ , and consequently on the curvatures which depend on these first derivatives. We first state an immediate corollary of Lemma 3, which controls  $|g_s|$ .

**Corollary 4.** *Solutions originating from initial data satisfying our Closeness Assumptions have  $|g_s| \leq 1$  for as long as they exist.*

*Proof.* Because, as noted above, the ordering  $f \leq g$  is preserved by the flow, it follows from Lemma 3 that  $f^2 g_s^2 \leq g^2 g_s^2 \leq f^2$ . The result follows.  $\square$

Next, we obtain a bound for  $|f_s|$ .

**Lemma 5.** *If  $f \leq g$  initially, then for as long as the flow exists,*

$$|f_s| \leq \max \left\{ \frac{2}{\sqrt{3}}, \max |f_s(\cdot, 0)| \right\}.$$

*Proof.* Using equation (21) of [IKS16] together with the fact that

$$\Delta \Omega = \Omega_{ss} + (f_s/f + 2g_s/g) \Omega_s$$

holds for any smooth function  $\Omega(s, t)$ , we see that  $f_s$  evolves by

$$(26) \quad (f_s)_t = (f_s)_{ss} + \left\{ 2 \frac{g_s}{g} - \frac{f_s}{f} \right\} (f_s)_s - \left\{ 6 \frac{f^2}{g^4} + 2 \frac{g_s^2}{g^2} \right\} f_s + 8 \frac{f^3}{g^5} g_s.$$

Because  $f_s \Big|_{\mathcal{S}_{\pm}^2} = \pm 1$ , we do not need to worry about a maximum of  $|f_s|$  on  $\mathcal{S}_{\pm}^2$ .

We apply the weighted Cauchy–Schwarz inequality  $|ab| \leq \epsilon a^2 + (1/4\epsilon)b^2$  to the term  $8f^3 g_s/g^5$  above, with  $a = g_s/g$  and  $b = f^3/g^4$ . Thus if  $(f_s)_{\max} = C > 0$  at some time, we obtain

$$\begin{aligned} \frac{d}{dt} (f_s)_{\max} &\leq -(f_s)_{\max} \left( 6 \frac{f^2}{g^4} + 2 \frac{g_s^2}{g^2} \right) + 8 \frac{f^3}{g^5} g_s \\ &\leq \left( \frac{4}{\sqrt{3}} - 2C \right) \frac{g_s^2}{g^2} + \left( \frac{12}{\sqrt{3}} \frac{f^4}{g^4} - 6C \right) \frac{f^2}{g^4} \\ &\leq 0 \end{aligned}$$

if  $C \geq 2/\sqrt{3}$ , because  $f/g \leq 1$ .

A similar argument shows that  $\frac{d}{dt}(f_s)_{\min} \geq 0$  if  $(f_s)_{\min} = -C$  at some time.  $\square$

These uniform bounds on the first derivatives of  $f$  and  $g$  lead to the following.

**Lemma 6.** *For any solution originating from initial data satisfying our Closeness Assumptions, there exists a uniform constant  $C$  such that*

$$|\kappa_{12}| + |\kappa_{31}| + |\kappa_{23}| \leq \frac{C}{g^2}$$

for as long as the flow exists.

*Proof.* It follows from Lemma 3 that the inequality  $(gg_s/f)^2 \leq 1$  persists if it is true initially. This implies that  $|g_s| \leq f/g$  for as long as the flow exists. Combining this estimate with the identities in (4), using Corollary 4, Lemma 5, and the fact that  $f \leq g$ , we obtain

$$|\kappa_{12}| + |\kappa_{31}| + |\kappa_{23}| \leq \frac{C}{g^2}.$$

$\square$

**4.3. Second derivative estimates.** Here we derive estimates for the remaining curvatures — those that depend on second-order derivatives of  $(f, g)$ .

**Lemma 7.** *For any solution originating from initial data satisfying our Closeness Assumptions, there exists a uniform constant  $C$  such that*

$$|\kappa_{02}| = \left| \frac{g_{ss}}{g} \right| \leq \frac{C}{g^2}$$

for as long as the flow exists.

*Proof.* We define  $Q = gg_{ss} - Ag_s^2 - Bf_s^2$ , where  $A, B > 0$  are to be suitably chosen below. We first show that there exists a uniform constant  $C$  so that  $Q \geq -C$  for as long as the flow exists. A straightforward computation shows that the evolution of  $Q$  is given by

$$\begin{aligned} (27) \quad \frac{\partial Q}{\partial t} = & \Delta Q + \frac{12Bf^2f_s^2}{g^4} + \frac{4f_s^2}{g^2} + \frac{24f^2g_s^2}{g^4} + \frac{12Af^2g_s^2}{g^4} + \frac{2Af_s^2g_s^2}{f^2} + \frac{4Bf_s^2g_s^2}{g^2} \\ & + \frac{2g_s^4}{g^2} + \frac{2Ag_s^4}{g^2} + 2Bf_{ss}^2 + 2(A-1)g_{ss}^2 - gg_{ss} \left( \frac{4f^2}{g^4} + \frac{2f_s^2}{f^2} + \frac{4Ag_s^2}{g^2} \right) \\ & - \frac{16Bf^3f_sg_s}{g^5} - \frac{24ff_sg_s}{g^3} - \frac{8Aff_sg_s}{g^3} + \frac{2gf_s^3g_s}{f^3} - \frac{8g_s^2}{g^2} - \frac{8Ag_s^2}{g^2} \\ & + \frac{4ff_{ss}}{g^2} + \frac{4Bf_s^2f_{ss}}{f} - \frac{8Bf_sg_sf_{ss}}{g} - \frac{2gf_sg_sf_{ss}}{f^2}, \end{aligned}$$

where as noted above,  $\Delta Q = Q_{ss} + (f_s/f + 2g_s/g)Q_s$ . We observe that l'Hôpital's rule implies that the terms

$$\frac{Q_sf_s}{f}, \quad \frac{2Af_s^2g_s^2}{f^2}, \quad 2gf_s^2 \frac{(f_sg_s - fg_{ss})}{f^3}, \quad \frac{4Bf_s^2f_{ss}}{f},$$

appearing in equation (27) are well defined and smooth at  $S_{\pm}^2$ . We now distinguish between two cases.

*Case 1.* A minimum of  $Q$  occurs away from  $S_{\pm}^2$ .

We assume that at a minimum of  $Q$  at some time  $t$ , we have  $gg_{ss} - Ag_s^2 - Bf_s^2 \leq -\bar{C}$  for a large constant  $\bar{C} > 0$  to be chosen. Because we are bounding  $Q$  from below, we may assume that  $g_{ss} \leq 0$ . Then since Corollary 4 and Lemma 5 give uniform bounds for  $|f_s|$  and  $|g_s|$ , we may choose  $\bar{C}$  sufficiently large relative to  $A$  and  $B$  such that

$$(28) \quad -gg_{ss} \left( \frac{4f^2}{g^4} + \frac{2f_s^2}{f^2} + \frac{4Ag_s^2}{g^2} \right) \geq \frac{\bar{C}f^2}{2g^4} + \frac{\bar{C}f_s^2}{2f^2} + \frac{\bar{C}Ag_s^2}{g^2}.$$

It then follows from (27) that at a minimum of  $Q$  at time  $t$ , we have

$$(29) \quad \begin{aligned} \frac{d}{dt}Q_{\min} &\geq 2Bf_{ss}^2 + \frac{\bar{C}f^2}{2g^4} + \frac{\bar{C}f_s^2}{2f^2} + \frac{\bar{C}Ag_s^2}{g^2} \\ &\quad - \frac{16Bf^3f_sg_s}{g^5} - \frac{24ff_sg_s}{g^3} - \frac{8Aff_sg_s}{g^3} + \frac{2gf_s^3g_s}{f^3} - \frac{8g_s^2}{g^2} - \frac{8Ag_s^2}{g^2} \\ &\quad + \frac{4ff_{ss}}{g^2} + \frac{4Bf_s^2f_{ss}}{f} - \frac{8Bf_sg_sf_{ss}}{g} - \frac{2gf_sg_sf_{ss}}{f^2}. \end{aligned}$$

To estimate the terms in (29) containing  $f_{ss}$ , we use Lemma 3, Corollary 4, Lemma 5, the facts that  $f \leq g$  and  $|g_s| \leq f/g$ , and a weighted Cauchy–Schwarz inequality to determine that there exists a uniform constant  $C'$  such that

$$\begin{aligned} &\left| \frac{4ff_{ss}}{g^2} \right| + \left| \frac{4Bf_s^2f_{ss}}{f} \right| + \left| \frac{8Bf_sg_sf_{ss}}{g} \right| + \left| \frac{2gf_sg_sf_{ss}}{f^2} \right| \\ &\leq \left( \frac{1}{2}f_{ss}^2 + \frac{C'f^2}{g^4} \right) + \left( \frac{B}{2}f_{ss}^2 + C'B\frac{f_s^2}{f^2} \right) + \left( \frac{B}{2}f_{ss}^2 + C'B\frac{g_s^2}{g^2} \right) + \left( \frac{1}{2}f_{ss}^2 + C'\frac{f_s^2}{f^2} \right) \\ &\leq (B+1)f_{ss}^2 + C'(B+1) \left( \frac{f_s^2}{f^2} + \frac{g_s^2}{g^2} + \frac{f^2}{g^4} \right). \end{aligned}$$

The remaining terms in (29) can be estimated in a similar manner. Thus we find that

$$\begin{aligned} \frac{d}{dt}Q_{\min} &\geq 2Bf_{ss}^2 + \frac{\bar{C}f^2}{2g^4} + \frac{\bar{C}f_s^2}{2f^2} + \frac{\bar{C}Ag_s^2}{g^2} \\ &\quad - C'(1+A+B) \left( \frac{f^2}{g^4} + \frac{f_s^2}{f^2} + \frac{g_s^2}{g^2} \right) - (B+1)f_{ss}^2 \\ &\geq 0, \end{aligned}$$

if we choose  $A = 1$ ,  $B = 2$  and  $\bar{C}$  sufficiently large so that  $\bar{C} > C'(1+A+B)$ . Therefore, in this case, either  $Q \geq -\bar{C}$  or  $\frac{d}{dt}Q_{\min} \geq 0$ .

*Case 2.* A minimum of  $Q$  occurs on  $\mathcal{S}_{\pm}^2$ .

The only difference from Case 1 is that one must deal with the term  $\frac{Q_s f_s}{f}$  at  $\mathcal{S}_{\pm}^2$ . We apply l'Hôpital's rule to see that

$$\frac{Q_s f_s}{f} \Big|_{\mathcal{S}_{\pm}^2} = \left( Q_{ss} + Q_s \frac{f_{ss}}{f_s} \right) \Big|_{\mathcal{S}_{\pm}^2} = Q_{ss} \Big|_{\mathcal{S}_{\pm}^2}.$$

However, smoothness of either function  $Q_{\pm}(s, \cdot) := Q(s - s_{\pm}, \cdot)$  at a minimum on  $\mathcal{S}_{\pm}^2$  shows that  $Q_{ss}|_{\mathcal{S}_{\pm}^2} = (Q_{\pm})_{ss}|_{\mathcal{S}_{\pm}^2} \geq 0$ . A similar computation as in Case 1 then yields

$$\frac{d}{dt}Q|_{\mathcal{S}_{\pm}^2} \geq 0,$$

unless  $Q_{\min}(t) = Q(\cdot, t)|_{\mathbb{S}^2_{\pm}} \geq -\bar{C}$ , for the constant  $\bar{C}$  chosen in Case 1.

Combined, Case 1 and Case 2 show that

$$Q(\cdot, t) \geq \min \{ -\bar{C}, Q_{\min}(0) \}.$$

In particular, this implies that

$$\frac{g_{ss}}{g} \geq -\frac{C}{g^2},$$

for a uniform constant  $C$  as long as the flow exists.

Finally, considering the quantity  $\tilde{Q} := gg_{ss} + Ag_s^2 + Bf_s^2$  and bounding  $\tilde{Q}$  from above using similar arguments yields a uniform constant  $C$  such that

$$\frac{g_{ss}}{g} \leq \frac{C}{g^2}$$

for as long as the flow exists. This concludes the proof of the Lemma.  $\square$

We now define

$$(30) \quad \mu(t) := \min_{\mathbb{S}^2 \times \mathbb{S}^2} g(\cdot, t),$$

observing that Lemmas 6 and 7 imply that there exists a uniform constant  $C$  such that as long as the flow exists, one has

$$(31) \quad |\kappa_{12}| + |\kappa_{13}| + |\kappa_{23}| + |\kappa_{02}| + |\kappa_{03}| \leq \frac{C}{\mu^2}.$$

**Remark 4.** Controlling the curvature  $\kappa_{01}$  is considerably more subtle. This is because, even for Kähler solutions, the alternative in statement (II) of Lemma 8 below is truly necessary: estimate (33) need not hold unless such solutions originate from initial data satisfying part (d) of our Closeness Assumptions. Solutions for which part (d) is false can have  $f \searrow 0$  uniformly as  $t \nearrow T$ , with  $g(\cdot, T) > 0$  everywhere. Each such (unrescaled) solution converges in the Gromov–Hausdorff sense to a  $\mathbb{CP}^1$  of multiplicity two; see Theorem 1.1 of [SW11].

**Lemma 8.** For any solution originating from initial data satisfying our Closeness Assumptions, the following are true:

(I) The sectional curvature  $\kappa_{01} = -f_{ss}/f$  satisfies

$$(32) \quad \kappa_{01} \geq -\frac{C}{g^2}$$

for a uniform constant  $C$ .

(II) Either there is an analogous upper bound

$$(33) \quad \kappa_{01} \leq \frac{C}{\mu^2},$$

or any finite-time singularity is Type-I.

*Proof.* Because the scalar curvature  $R$  is a supersolution of the heat equation (in the sense that  $(\partial_t - \Delta)R \geq 0$ ), there exists a constant  $r_0$  depending only on the initial data such that for as long as the flow exists, one has

$$r_0 \leq R = \kappa_{01} + \kappa_{02} + \kappa_{03} + \kappa_{12} + \kappa_{23} + \kappa_{31},$$

where  $\kappa_{02} = \kappa_{03}$ . Using this together with Lemma 6, Lemma 7, and the fact that  $\frac{d}{dt}g_{\max} \leq 0$ , we get the lower bound (32).

To prove (II), we assume that (33) fails and use a blow-up argument at a finite-time singularity. In particular, we assume that  $T < \infty$  is a singular time for the flow, and that

$$(34) \quad \limsup_{t \rightarrow T} \left( \sup_{S^2 \times S^2} \kappa_{01}(\cdot, t) \mu(t)^2 \right) = \infty.$$

We now let  $t_i \rightarrow T$  as  $i \rightarrow \infty$  such that

$$\sup_{t \in [0, t_i]} \left( \sup_{S^2 \times S^2} \kappa_{01}(\cdot, t) \mu(t)^2 \right) = \kappa_{01}(p_i, t_i) \mu(t_i)^2$$

for some  $p_i \in M$ , and we let  $K_i := \kappa_{01}(p_i, t_i)$ . It follows from our choice of  $t_i$  that

$$(35) \quad K_i \mu(t_i)^2 \rightarrow \infty \quad \text{as } i \rightarrow \infty.$$

We define the blow-up sequence of solutions  $G$  of metric form (1) as follows:

$$G_i(\cdot, t) := K_i G(\cdot, t_i + t K_i^{-1}), \quad (-K_i t_i \leq t < (T - t_i) K_i).$$

We claim that the curvatures of the rescaled metrics  $G_i$  are uniformly bounded. To prove the claim for  $\kappa_{12}$ , say, we begin by noting that estimate (31) implies that

$$(36) \quad |\kappa_{12}^i(\cdot, t)| = \frac{|\kappa_{12}(\cdot, t_i + t K_i^{-1})|}{K_i} \leq \frac{C}{K_i \mu(t_i + t K_i^{-1})^2}.$$

It follows from Remark 1 of [IKS16] that the evolution equation for  $g(\cdot, t)$  can be written as

$$\frac{\partial}{\partial t} (\log g) = -\kappa_{02} - \kappa_{23} - \kappa_{31},$$

which implies that

$$\left| \frac{\partial}{\partial t} \log g \right| \leq \frac{C}{g^2},$$

and therefore that

$$\left| \frac{d}{dt} \mu^2 \right| \leq C.$$

Integrating this over  $[t_i + t K_i^{-1}, t_i]$  yields

$$|\mu(t_i + t K_i^{-1})^2 - \mu(t_i)^2| \leq \frac{C}{K_i},$$

for, say,  $t \in [-1, 0]$ . This implies that

$$\mu(t_i + t K_i^{-1})^2 \geq \mu(t_i)^2 - \frac{C}{K_i},$$

whereupon (36) implies for  $t \in [-1, 0]$  that

$$|\kappa_{12}^i(\cdot, t)| \leq \frac{C}{K_i \mu(t_i)^2 - C} \rightarrow 0$$

as  $i \rightarrow \infty$ , because (35) holds.

To bound the remaining curvatures of the rescaled metrics, we use similar arguments together with (31) to conclude that

$$|\kappa_{12}^i(\cdot, t)| + |\kappa_{13}^i(\cdot, t)| + |\kappa_{23}^i(\cdot, t)| + |\kappa_{02}^i(\cdot, t)| + |\kappa_{03}^i(\cdot, t)| \leq \frac{C}{K_i \mu(t_i)^2 - C} \rightarrow 0$$



as  $i \rightarrow \infty$ , and we use (32) to show that

$$\kappa_{01} \geq -\frac{C}{K_i \mu(t_i)^2 - C} \rightarrow 0,$$

as  $i \rightarrow \infty$ .

After extracting a convergent subsequence, we determine that  $(S^2 \tilde{\times} S^2, G_i(t), p_i)$  converges in the pointed Cheeger–Gromov–Hamilton sense to a complete ancient solution

$$(\mathcal{M}_\infty^4, \mathcal{G}_\infty(t), p_\infty)$$

that exists for  $t \in (-\infty, t^*)$  where  $t^* := \lim_{i \rightarrow \infty} (T - t_i) K_i \leq \infty$ . Moreover, one has

$$(37) \quad \kappa_{12}^\infty = \kappa_{13}^\infty = \kappa_{23}^\infty = \kappa_{02}^\infty = 0, \quad \text{and} \quad \kappa_{01}^\infty \geq 0,$$

with  $\kappa_{01}(p_\infty, 0) = 1$ . By applying Hamilton’s splitting theorem [Ham93] twice, we find that the universal cover  $(\tilde{\mathcal{M}}_\infty^4, \mathcal{G}_\infty, p_\infty)$  splits isometrically as the product of  $\mathbb{R}^2$  and a complete ancient solution  $(N^2, \mathcal{G}_\infty|_{N^2})$  with bounded positive scalar curvature. It follows from the classification in [DHS12] and [DS06] that  $(N^2, \mathcal{G}_\infty|_{N^2})$  is either the King–Rosenau solution, the cigar, or the round sphere  $S^2$ . In the former case, it is a standard fact that by choosing a modified sequence  $\tilde{p}_i$  of blow-up points, one can obtain the cigar as a limit. But this is impossible by Perelman’s  $\kappa$ -non-collapsing result [Per02]. So the limit must be isometric to one of the products  $S^2 \times \mathbb{R}^2$  or  $\mathbb{R}^2 \times S^2$ . In either case,<sup>3</sup> the singularity is Type-I, and we have  $\kappa_{01} \leq C/(T - t)$ .  $\square$

## 5. SINGULARITY FORMATION

Our main results in this section show that for Ricci flow solutions originating from initial data satisfying our Closeness Assumptions, all singularities are Type-I, with  $|S_-^2| = 0$  at the singular time  $T < \infty$ . This requires some work, for the following reason. Away from the special fibers  $S_\pm^2$ , the geometry of  $(S^2 \tilde{\times} S^2, G)$  is that of  $(a, b) \times S^3$ . So without appropriate assumptions on the initial data, it is highly plausible that neckpinch singularities like those analyzed in [IKS16] could develop at a fiber  $\{s_0\} \times S^3$  far from  $S_-^2$ . Our results in this section rule out this possibility for solutions originating from initial data satisfying our Closeness Assumptions.

As shown in [SW11] and as noted above, the behavior of Kähler solutions depends strongly on whether  $|S_+^2| < 3|S_-^2|$ ,  $|S_+^2| = 3|S_-^2|$ , or  $|S_+^2| > 3|S_-^2|$ . It follows from part (d) of our Closeness Assumptions that the solutions we study have  $|S_+^2| > 3|S_-^2|$  initially. Our first result in this section shows that this threshold condition is preserved by the flow, even for non-Kähler solutions, provided they originate from initial data satisfying the Closeness Assumptions.

**Lemma 9.** *Solutions originating from initial data satisfying our Closeness Assumptions satisfy*

$$g^2(s_+, t) - 3g^2(s_-, t) \geq \delta^2$$

*for as long as they exist.*

<sup>3</sup>For the metrics we study here, the case  $S^2 \times \mathbb{R}^2$  corresponds to the  $g^2(\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3)$  factor becoming flat after rescaling, while the case  $\mathbb{R}^2 \times S^2$  corresponds to the  $(ds \otimes ds + f^2 \omega^1 \otimes \omega^1)$  factor becoming flat.

*Proof.* We recall that

$$g_t = g_{ss} + \left( \frac{f_s}{f} + \frac{g_s}{g} \right) g_s + 2 \frac{(f^2 - 2g^2)}{g^3}.$$

Using l'Hôpital's rule, we compute at  $s_+$  that

$$\lim_{s \rightarrow s_+} \frac{f_s g_s}{f} = g_{ss}(s_+, t).$$

Because  $g_s(s_+, t) = 0$ , we have

$$\frac{d}{dt} g(s_+, t) = 2g_{ss}(s_+, t) - \frac{4}{g(s_+, t)}.$$

Lemma 3 tells us that  $g|g_s| \leq f$ , which as a consequence of l'Hôpital's rule, implies at  $s_+$  that

$$gg_{ss} \geq -1.$$

It follows that

$$(38) \quad \frac{d}{dt} g^2(s_+, t) \geq -12.$$

Similarly, using the fact that  $g|g_s| \leq f$  and using l'Hôpital's rule, we have that  $gg_{ss} \leq 1$  at  $s_-$ , from which we obtain

$$(39) \quad \frac{d}{dt} g^2(s_-, t) \leq -4.$$

Estimates (38) and (39) together imply that

$$\frac{d}{dt} (g^2(s_+, t) - 3g^2(s_-, t)) \geq 0,$$

which yields

$$g^2(s_+, t) - 3g^2(s_-, t) \geq g^2(s_+, 0) - 3g^2(s_-, 0).$$

□

Our second result in this section proves that solutions originating from initial data satisfying our Closeness Assumptions become singular at  $T < \infty$  only if  $g$  vanishes somewhere.

**Lemma 10.** *If a solution originating from initial data satisfying our Closeness Assumptions becomes singular at time  $T$ , then  $\mu(T) = 0$ .*

*Proof.* Lemma 8 proves that either there is a two-sided curvature bound for  $\kappa_{01}$  or the singularity is Type-I.

If there is a two-sided bound  $|\kappa_{01}| \leq C/\mu^2$ , then combining this with estimate (31) we obtain a uniform constant  $C$  such that

$$|\text{Rc}(G(t))| \leq \frac{C}{\mu^2},$$

for as long as the flow exists. Because [Ses05] proves that  $\limsup_{t \nearrow T} |\text{Rc}| = \infty$  if  $T < \infty$  is the singularity time, it follows that  $\mu(T) = 0$ .

To complete the proof, we may assume, to obtain a contradiction, that a solution encounters a finite-time Type-I singularity for which  $\lim_{t \rightarrow T} \mu(t) = 0$  is false.

We first claim that this assumption implies that there exists  $\eta > 0$  such that  $\mu(t) \geq \eta > 0$  for  $t \in [0, T)$ . We prove this claim by contradiction. Observe that the maximum principle implies that

$$\frac{d}{dt}\mu(t) \geq -\frac{4}{\mu(t)}.$$

So for  $t \geq \tau$  in  $[0, T)$ , one has

$$(40) \quad \mu(t)^2 \geq \mu(\tau)^2 - 8(t - \tau).$$

If it is not true that  $\lim_{t \rightarrow T} \mu(t) = 0$ , then there exists a sequence  $\tau_i \rightarrow T$  along which  $\mu(\tau_i) \geq \eta > 0$  for all  $i$ . On the other hand, if there exists another sequence  $t_i \rightarrow T$  along which  $\lim_{i \rightarrow \infty} \mu(t_i) = 0$ , then by passing to subsequences, we may assume that  $t_i \geq \tau_i$ , and hence that

$$\mu(t_i)^2 \geq \mu(\tau_i)^2 - 8(t_i - \tau_i) \geq \eta^2 - 8(t_i - \tau_i).$$

But this is impossible, because  $\lim_{i \rightarrow \infty} \mu(t_i) = 0$  and  $\lim_{i \rightarrow \infty} (t_i - \tau_i) = 0$ . This contradiction proves the claim.

The proof of Lemma 8 tells us that the inequality  $\mu(t) \geq \eta > 0$  implies that the universal cover of any Type-I singularity model must be  $\mathbb{S}^2 \times \mathbb{R}^2$ . Compactness of the  $\mathbb{S}^2$  factor implies there is a sequence  $t_i \rightarrow T$  along which  $\sup_{s_- \leq s \leq s_+} f(s, t_i) \leq C\sqrt{T - t_i}$ . On the other hand it follows from Lemma 3 that

$$g|g_s| \leq f \leq C\sqrt{T - t_i}$$

at those times, which implies that

$$(41) \quad g^2(s_+, t_i) - g^2(s_-, t_i) \leq C\sqrt{T - t_i}(s_+ - s_-).$$

We recall that

$$\frac{d}{dt}(s - s_-) = \int_{s_-}^s \left( \frac{f_{ss}}{f} + 2\frac{g_{ss}}{g} \right) ds = - \int_{s_-}^s (\kappa_{01} + 2\kappa_{02}) ds.$$

Combining Lemma 7 and part (I) of Lemma 8, we obtain

$$\frac{d}{dt}(s - s_-) \leq \frac{C(s - s_-)}{\mu(t)^2} \leq C'(s - s_-),$$

because  $\mu(t) \geq \eta > 0$ . Integrating this over  $[0, T)$  yields a constant  $C''$  such that

$$(42) \quad |s - s_-| \leq C'' \quad \text{for all } t \in [0, T) \quad \text{and } s \in [s_-, s_+].$$

Combining (41) and (42) then gives us

$$g^2(s_+, t_i) - g^2(s_-, t_i) \leq C\sqrt{T - t_i}.$$

But this is incompatible with the conclusion of Lemma 9 that

$$g^2(s_+, t_i) - 3g^2(s_-, t_i) \geq \delta^2 > 0.$$

This contradiction proves the result.  $\square$

Although we are not able to prove that part (e) of our Closeness Assumptions—monotonicity in space of  $g(s, 0)$ —is preserved, we are able to show that  $g$  is trapped above a lower barrier that is monotone increasing in space. Consequently the spatial minimum  $\mu(t)$  of  $g(s, t)$  must occur at  $s_-$  for all time.

**Lemma 11.** *If a solution originating from initial data satisfying our Closeness Assumptions exists for  $t \in [0, T)$ , then*

$$\mu(t) = g(s_-, t)$$

for all  $0 \leq t \leq T$ .

*Proof.* We define the constant  $c = 4 + 2/\sqrt{3}$  and the function

$$H = g^2 + 2ct.$$

We claim that  $H$  is a supersolution of the heat equation. To prove the claim, we recall that it follows from Lemma 3 that  $g|g_s| \leq f$ . Using this together with Lemma 5, we estimate that

$$\begin{aligned} gg_t &= gg_{ss} + g_s^2 + \frac{gg_s}{f} f_s + 2\frac{f^2}{g^2} - 4 \\ &\geq gg_{ss} + g_s^2 - |f_s| - 4 \\ &\geq gg_{ss} + g_s^2 - c. \end{aligned}$$

It follows that

$$H_t - H_{ss} = (g^2)_t - (g^2)_{ss} + 2c \geq 0,$$

which proves the claim.

We now let  $h$  be a solution of the linear heat equation,  $h_t = h_{ss}$ , with initial condition

$$h(s, 0) = H(s, 0)$$

and with boundary values

$$h(s_-, t) = H(s_-, t), \quad h(s_+, t) = H(s_+, t).$$

Then the maximum principle implies that  $H \geq h$  for as long as both functions exist.

Finally, by construction and as a consequence of part (e) of our Closeness Assumptions, we observe that  $h_s(s, 0) = H_s(s, 0) = 2gg_s(s, 0) \geq 0$ , with strict inequality off  $S_{\pm}^2$ . We use the commutator (6) to compute that

$$(h_s)_t = (h_s)_{ss} + \left( \frac{f_{ss}}{f} + \frac{2g_{ss}}{g} \right) h_s.$$

For each  $\varepsilon > 0$ , because the metric  $G(t)$  is smooth for  $t \in [0, T - \varepsilon]$ , there exists a constant  $C_\varepsilon$  such that

$$(h_s)_t \geq (h_s)_{ss} - C_\varepsilon h_s, \quad (0 \leq t \leq T - \varepsilon).$$

Applying the maximum principle on each time interval and then letting  $\varepsilon \searrow 0$ , we obtain  $h_s \geq 0$  for  $s \in [s_-, s_+]$  and  $t \in [0, T)$ , and by continuity, for  $t = T$  as well. This implies that  $g^2$  is bounded from below by a barrier,  $h - 2ct$ , that is monotone increasing in space at each time. Because

$$g^2(s_-, t) = h(s_-, t) - 2ct = \min_{s_- \leq s \leq s_+} \{h(s, t) - 2ct\} \leq \mu(t),$$

the result follows.  $\square$

Combining this result with Lemma 10, we determine that solutions originating from initial data satisfying our Closeness Assumptions become singular only by crushing the fiber  $S_-^2$ . We state this as follows:

**Corollary 12.** *If a solution originating from initial data satisfying our Closeness Assumptions becomes singular at time  $T$ , then  $g(s_-, T) = 0$ .*

**Corollary 13.** *All solutions originating from initial data satisfying our Closeness Assumptions develop finite-time Type-I singularities.*

*Proof.* It follows from Lemma 11 and from estimate (39) that

$$(43) \quad \frac{d}{dt}(\mu^2(t)) \leq -4.$$

So a finite-time singularity is inevitable. As a consequence of Lemma 8, to prove that the singularity is Type-I, we may assume there is a two-sided curvature bound for  $\kappa_{01}$ . Such a bound, together with estimate (31), gives a uniform constant  $C$  such that  $|\text{Rc}(G(t))| \leq C\mu^{-2}(t)$  for as long as the flow exists. But then the result follows easily from estimate (43).  $\square$

## 6. CONVERGENCE TO THE BLOWDOWN SOLITON

Corollaries 12 and 13 tell us that any point  $p \in \mathcal{S}_-^2$  is a *special Type-I singular point* in the sense of Enders–Müller–Topping [EMT11]. It follows from that work that every blow-up sequence  $(\mathcal{S}_-^2 \times \mathcal{S}_-^2, G_k(t), p)$  subconverges to a smooth nontrivial gradient shrinking soliton  $(\mathcal{M}, G_\infty(\tau))$  defined for  $-\infty < \tau < 0$ . Using Lemma 9, we determine that the limit is noncompact. So  $\mathcal{M}$  is diffeomorphic to  $\mathbb{C}^2$  blown up at the origin; that is,  $\mathcal{O}(-1)$ . Moreover, the symmetries of  $G(t)$  are preserved in the limit, so the metric retains the form exhibited in (1):

$$(44) \quad G_\infty = ds \otimes ds + \left\{ f^2 \omega^1 \otimes \omega^1 + g^2 (\omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3) \right\}.$$

Here and in the remainder of this section, we abuse notation by using  $s$  for the soliton coordinate, and using  $f$  and  $g$  for the components of the limit soliton metric.

The quantity  $\psi$  that we estimate in Lemma 3 is scale-invariant, so the limit soliton satisfies

$$(45) \quad -1 \leq \frac{gg_s}{f} \leq 1,$$

which implies that the limit is “not too far” from Kähler in a precise sense. It is a general principle that shrinking solitons appear in discrete rather than continuous families, modulo scaling and isometry. So it is reasonable to conjecture that there are no other cohomogeneity-one shrinking solitons in the neighborhood of such metrics satisfying estimate (45). That “gap conjecture” is true, as we now demonstrate. (For a related rigidity result, see recent work [Kot17] of Kotschwar.)

**Lemma 14.** *Any smooth gradient shrinking soliton  $(\mathcal{M}, G_\infty(\tau))$  having the form (44) and satisfying estimate (45) is Kähler.*

*Proof.* We work at a fixed time  $\tau < 0$  and so suppress time below. However, we continue to use subscripts to indicate spatial derivatives. We note here that smoothness requires that the closing conditions (2) hold at  $s = 0$ , a fact we use freely below.

We define

$$F(s) = f - gg_s.$$

We have  $F(0) = 0$ . The estimate (45) tells us that this is a global minimum. So  $F_s(0) = 0$  as well.

We now let  $\Gamma$  denote the soliton potential function, and we set  $\gamma = \Gamma_s$ . Using equation (51) from [IKS16] to compute the Lie derivative, we find that the soliton equation

$$(46) \quad -\text{Rc}[G_\infty] = \lambda G_\infty + \frac{1}{2} \mathcal{L}_{\nabla \Gamma} G_\infty$$

becomes the system

$$(47a) \quad \gamma_s = \frac{f_{ss}}{f} + 2 \frac{g_{ss}}{g} - \lambda,$$

$$(47b) \quad \frac{f_{ss}}{f} = \frac{f_s \gamma}{f} - 2 \frac{f_s g_s}{f g} + 2 \frac{f^2}{g^4} + \lambda,$$

$$(47c) \quad \frac{g_{ss}}{g} = \frac{g_s \gamma}{g} - \frac{f_s g_s}{f g} - \frac{g_s^2}{g^2} - 2 \frac{f^2}{g^4} + \frac{4}{g^2} + \lambda,$$

where  $\lambda < 0$  depends on our choice of  $\tau$  above.

Computing  $F_s$  using equation (47c), one finds that

$$(48) \quad \begin{aligned} F_s &= f_s - g_s^2 - g g_{ss} \\ &= f_s - g g_s \gamma + \frac{g f_s g_s}{f} + 2 \frac{f^2}{g^2} - \lambda g^2 - 4 \\ &= \left( \gamma - \frac{f_s}{f} \right) F + 2 f_s - f \gamma + 2 \frac{f^2}{g^2} - \lambda g^2 - 4. \end{aligned}$$

Hence

$$F_{ss} = \left( \gamma - \frac{f_s}{f} \right) F_s + \left( \gamma - \frac{f_s}{f} \right)_s F + X,$$

where we use (47b) to rewrite the final term above as

$$\begin{aligned} X &= 2 f_{ss} - f_s \gamma - f \gamma_s + 4 \left( \frac{f f_s}{g^2} - \frac{f^2 g_s}{g^3} \right) - 2 \lambda g g_s \\ &= \left( 4 \frac{f^2}{g^4} + 4 \frac{f_s}{g^2} + 2 \lambda \right) F + \gamma^2 \left( \frac{f}{\gamma} \right)_s. \end{aligned}$$

Therefore,  $F$  satisfies the linear second-order (seemingly inhomogeneous) ODE

$$(49) \quad F_{ss} + \left( \frac{f_s}{f} - \gamma \right) F_s - \left\{ \left( \frac{f_s}{f} - \gamma \right)_s + 4 \frac{f_s}{g^2} + 4 \frac{f^2}{g^4} + 2 \lambda \right\} F = \gamma^2 \left( \frac{f}{\gamma} \right)_s.$$

We now show that the term on the RHS can be rewritten in terms of  $F$  and  $F_s$ . Using equations (47a), (47b), and (47c) in order, and then applying the identity

$gg_s = f - F$ , we obtain

$$\begin{aligned}
 \frac{1}{2}\gamma^2 \left( \frac{f}{\gamma} \right)_s &= \frac{1}{2}(f_s\gamma - f\gamma_s) \\
 &= \frac{1}{2} \left( f_s\gamma - f_{ss} - 2\frac{fg_{ss}}{g} + \lambda f \right) \\
 &= \frac{f_sg_s}{g} - \frac{f^3}{g^4} - \frac{fg_{ss}}{g} \\
 &= 2\frac{f_sg_s}{g} - \frac{fg_s\gamma}{g} + \frac{fg_s^2}{g^2} + \frac{f^3}{g^4} - 4\frac{f}{g^2} - \lambda f \\
 (50) \quad &= - \left( 2\frac{f_s}{g^2} + \frac{f^2}{g^4} + \frac{fgg_s}{g^4} - \frac{f\gamma}{g^2} \right) F + Y,
 \end{aligned}$$

where

$$Y = 2\frac{ff_s}{g^2} + 2\frac{f^3}{g^4} - 4\frac{f}{g^2} - \frac{f^2\gamma}{g^2} - \lambda f.$$

Using equation (48) to rewrite the first term on the RHS, it is easy to see that

$$(51) \quad Y = \frac{f}{g^2}F_s + \frac{f_s - f\gamma}{g^2}F.$$

So by using equations (50) and (51), we find that equation (49) can be rewritten as the linear second-order *homogeneous* ODE

$$F_{ss} + \left( \frac{f_s}{f} - \gamma - 2\frac{f}{g^2} \right) F_s + \left\{ - \left( \frac{f_s}{f} - \gamma \right)_s + 2\frac{fgg_s}{g^4} - 2\frac{f_s}{g^2} - 2\frac{f^2}{g^4} - 2\lambda \right\} F = 0.$$

Because  $f_s/f \sim 1/s$  and  $-(f_s/f)_s \sim 1/s^2$  as  $s \searrow 0$ , this ODE has a regular singular point at  $s = 0$ . It is approximated in a neighborhood of  $s = 0$  by the equidimensional Euler equation  $s^2y(s) + sy'(s) + y(s) = 0$ , for which a fundamental set of solutions is  $\{\cos(\log s), \sin(\log s)\}$ . It then follows from a theorem of Frobenius that a fundamental set of solutions of the exact equation has the form

$$\sum_{n=0}^{\infty} a_n s^n [\cos(\log s)] \quad \text{and} \quad \sum_{n=0}^{\infty} b_n s^n [\sin(\log s)],$$

where all coefficients except  $a_0$  and  $b_0$  are determined by recurrence relations. We conclude that  $F$  is identically zero for all  $s \geq 0$ , hence that the soliton is Kähler.  $\square$

Theorem 1.5 of [FIK03] tells us that the blowdown soliton is unique up to scaling and isometry among  $U(2)$ -invariant Kähler–Ricci solitons. Hence this completes our proof of the Main Theorem.

## REFERENCES

- [BS09] **Brendle, Simon; Schoen.** Richard Manifolds with  $1/4$ -pinched curvature are space forms. *J. Amer. Math. Soc.* **22** (2009), no. 1, 287–307.
- [Cal82] **Calabi, Eugenio.** Extremal Kähler metrics. *Seminar on differential geometry* (S.-T. Yau, editor), Princeton University Press (1982) 259–290.
- [Cao96] **Cao, Huai-Dong.** Existence of gradient Kähler–Ricci solitons, *Elliptic and parabolic methods in geometry* (B. Chow, R. Gulliver, S. Levy, J. Sullivan, editors), AK Peters (1996) 1–16.
- [Cao97] **Cao, Huai-Dong.** Limits of solutions to the Kähler–Ricci flow, *J. Differential Geom.* **45** (1997) 257–272.

- [CHI04] **Cao, Huai-Dong; Hamilton, Richard S.; Ilmanen, Tom.** Gaussian densities and stability for some Ricci solitons. [arXiv:math/0404165v1](#).
- [CM12] **Colding, Tobias Holck; Minicozzi, William P., II.** Generic mean curvature flow I; generic singularities. *Ann. of Math.* **175** (2012), 755–833.
- [CM15] **Colding, Tobias Holck; Minicozzi, William P., II.** Uniqueness of blowups and Łojasiewicz inequalities. *Ann. of Math. (2)* **182** (2015), no. 1, 221–285.
- [DHS12] **Daskalopoulos, Panagiota; Hamilton, Richard; Šešum, Nataša.** Classification of ancient compact solutions to the Ricci flow on surfaces. *J. Differential Geom.* **91** (2012), no. 2, 171–214.
- [DS06] **Daskalopoulos, Panagiota; Šešum, Nataša.** Eternal solutions to the Ricci flow on  $\mathbb{R}^2$ . *Int. Math. Res. Not.* (2006) Art. ID 83610.
- [EMT11] **Enders, Joerg; Müller, Reto; Topping, Peter M.** On type-I singularities in Ricci flow. *Comm. Anal. Geom.* **19** (2011), no. 5, 905–922.
- [FIK03] **Feldman, Mikhail; Ilmanen, Tom; Knopf, Dan.** Rotationally symmetric shrinking and expanding gradient Kähler–Ricci solitons. *J. Differential Geom.* **65** (2003), no. 2, 169–209.
- [HHS14] **Hall, Stuart; Haslhofer, Robert; Siepmann, Michael.** The stability inequality for Ricci-flat cones. *J. Geom. Anal.* **24** (2014), no. 1, 472–494.
- [HM11] **Hall, Stuart J.; Murphy, Thomas.** On the linear stability of Kähler–Ricci solitons. *Proc. Amer. Math. Soc.* **139** (2011), no. 9, 3327–3337.
- [Ham93] **Hamilton, Richard S.** The formation of singularities in the Ricci flow. *Surveys in differential geometry, Vol. II* (Cambridge, MA, 1993), 7–136, Internat. Press, Cambridge, MA, 1995.
- [HN15] **Haslhofer, Robert; Naber, Aaron.** Weak solutions for the Ricci flow I. [arXiv:1504.00911v1](#).
- [IKS16] **Isenberg, James; Knopf, Dan; Šešum, Nataša.** Ricci flow neckpinches without rotational symmetry. *Comm. Partial Differential Equations*. In press. DOI:10.1080/03605302.2016.1233982.
- [Ive93] **Ivey, Thomas.** Ricci solitons on compact three-manifolds. *Differential Geom. Appl.* **3** (1993), no. 4, 301–307.
- [Koi90] **Koiso, Norihito.** On rotationally symmetric Hamilton’s equation for Kähler–Einstein metrics, *Recent Topics in Differential and Analytic Geometry* (T. Ochiai, editor), Advanced Studies in Pure Math **18**-1, Kinokuniya and Academic Press (1990) 327–337.
- [Kot17] **Kotschwar, Brett.** Kählerity of shrinking gradient Ricci solitons asymptotic to Kähler cones. [arXiv:1701.08486v1](#).
- [Kro13] **Kröncke, Klaus.** Ricci flow, Einstein metrics and the Yamabe invariant. [arXiv:1312.2224v1](#).
- [Max14] **Máximo, Davi.** On the blow-up of four-dimensional Ricci flow singularities. *J. Reine Angew. Math.* **692** (2014), 153–171.
- [Per02] **Perelman, Grisha.** The entropy formula for the Ricci flow and its geometric applications. [arXiv:math.DG/0211159](#).
- [Ses05] **Šešum, Nataša.** Curvature tensor under the Ricci flow. *Amer. J. Math.* **127** (2005), no. 6, 1315–1324.
- [SW11] **Song, Jian; Weinkove, Ben.** The Kähler–Ricci flow on Hirzebruch surfaces. *J. Reine Angew. Math.* **659** (2011), 141–168.
- [ST11] **Streets, Jeffrey; Tian, Gang.** Hermitian curvature flow. *J. Eur. Math. Soc. (JEMS)* **13** (2011), no. 3, 601–634.
- [Stu16] **Sturm, Karl-Theodor.** Super-Ricci flows for metric measure spaces. I. [arXiv:1603.02193v1](#).



(James Isenberg) UNIVERSITY OF OREGON

*E-mail address:* `isenberg@uoregon.edu`

*URL:* `http://www.uoregon.edu/~isenberg/`

(Dan Knopf) UNIVERSITY OF TEXAS AT AUSTIN

*E-mail address:* `danknopf@math.utexas.edu`

*URL:* `http://www.ma.utexas.edu/users/danknopf`

(Nataša Šešum) RUTGERS UNIVERSITY

*E-mail address:* `natasas@math.rutgers.edu`

*URL:* `http://www.math.rutgers.edu/~natasas/`